An Improvement on Vizing's Conjecture

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Abstract

Let $\gamma(G)$ denote the domination number of a graph G. A *Roman domination function* of a graph G is a function $f : V \rightarrow \{0, 1, 2\}$ such that every vertex with 0 has a neighbor with 2. The *Roman domination number* $\gamma_R(G)$ is the minimum of $f(V(G)) = \Sigma_{v\in V} f(v)$ over all such functions. Let $G\Box H$ denote the Cartesian product of graphs G and H. We prove that $\gamma(G)\gamma(H) \leq \gamma_R(G\Box H)$ for all simple graphs G and H, which is an improvement of $\gamma(G)\gamma(H) \leq 2\gamma(G\Box H)$ given by Clark and Suen [1], since $\gamma(G\Box H) \leq \gamma_R(G\Box H) \leq 2\gamma(G\Box H)$.

Key words: Vizing's Conjecture, domination number, Roman domination number

1 Introduction

In this note, we consider simple finite graphs only and follow [4] for terminology and definitions.

let $G = (V, E)$ be a graph with vertex set V and edge set E. For any vertex $v \in V$, the *open neighborhood of* v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood is $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighborhood is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A domination set of cardinality $\gamma(G)$ is called a γ-*set* of G. Recently, a variant of the domination number—Roman domination number is suggested by Stewart [5]. A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(u) = 2$. The weight of f is $f(V(G)) = \Sigma_{v\in V} f(v)$. The *Roman domination number*, denoted by $\gamma_R(G)$, equals the

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minimum weight of an RDF of G, and we say that a function f is a $\gamma_R(G)$-*function* if it is an RDF and $f(V(G)) = \gamma_R(G)$. For a graph G, let $f : V \to \{0, 1, 2\}$, and let (V_0, V_1, V_2) be the order partition of V induced by f, where $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = 0, 1, 2$. Note that there exists a 1-1 correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of $V(G)$. Thus we will write $f = (V_0, V_1, V_2)$.

Cockayne et al. [2] showed the following results.

Lemma 1. ([2]) *For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

Lemma 2. ([2]) *Let $f = (V_0, V_1, V_2)$ be any $\gamma_R(G)$-function. Then V_2 is a γ-set of $G[V_0 \cup V_2]$.*

For a pair of graphs G and H, the Cartesian product $G \Box H$ of G and H is the graph with vertex set $V(G) \times V(H)$ and where two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. In 1963, V. G. Vizing [6] conjectured the following:

Vizing's Conjecture. For any graphs G and H, $\gamma(G)\gamma(H) \leq \gamma(G \Box H)$.

We note that there are graphs G and H for which the above equality holds. The reader is referred to Hartnell and Rall [3] for a summary of recent progress on Vizing's conjecture. Recently, Clark and Suen [1] gave the following result.

Theorem 1. ([1]) *For any graphs G and H, $\gamma(G)\gamma(H) \leq 2\gamma(G \Box H)$.*

We shall show in this note that $\gamma(G)\gamma(H) \leq \gamma_R(G \Box H)$, which is an improvement of $\gamma(G)\gamma(H) \leq 2\gamma(G \Box H)$ by Lemma 1.

2 Main results

Theorem 2. *For any graphs G and H,*

$$\gamma(G)\gamma(H) \leq \gamma_R(G \Box H).$$

Proof. Let $f = (V_0, V_1, V_2)$ be any $\gamma_R(G \Box H)$-function of graph $G \Box H$. Denote $D = V_1 \cup V_2$. By Lemma 2, D and V_2 are domination set of graphs $G \Box H$ and $G \Box H - V_1$, respectively. Let $\{u_1, u_2, \ldots, u_{\gamma(G)}\}$ be a dominating set of G. Then we partition $V(G)$ into $\gamma(G)$ sets $\{\Pi_1, \Pi_2, \ldots, \Pi_{\gamma(G)}\}$ satisfying the following properties:

(i) $u_i \in \Pi_i$,

(ii) $u \in \Pi_i$ implies $u = u_i$ or u is adjacent to u_i.

Note that this partition is not unique. The partition of $V(G)$ induces a partition $\{D_1, D_2, \ldots, D_{\gamma(G)}\}$ of D where

$$D_i = (\Pi_i \times V(H)) \cap D.$$

Let P_i be the projection of D_i onto H. Then

$$P_i = \{v \mid (u, v) \in D_i \text{ for some } u \in \Pi_i\}.$$

For any i, $P_i \cup (V(H) - N_H[P_i])$ is a dominating set of H, so the number of vertices in $V(H)$ not dominated by P_i satisfies the inequality

$$|V(H) - N_H[P_i]| \geq \gamma(H) - |P_i|. \tag{1}$$

For $v \in V(H)$, denote

$$Q_v = V_2 \cap (V(G) \times \{v\}) = \{(u, v) \in V_2 \mid u \in V(G)\},$$

let C be the subset of $\{1, 2, \ldots, \gamma(G)\} \times V(H)$ given by

$$C = \{\, (i, v) \mid \Pi_i \times \{v\} \subseteq N_{G\Box H}[\, Q_v \,] \,\}.$$

Set

$$\begin{aligned} L_i &= \{(i, v) \in C \mid v \in V(H)\}, \\ R_v &= \{(i, v) \in C \mid 1 \leq i \leq \gamma(G)\}. \end{aligned}$$

It is clear that

$$N = |C| = \sum_{i=1}^{\gamma(G)} |L_i| = \sum_{v \in V(H)} |R_v|.$$

If $v \in V(H) - N_H[P_i]$, then the vertices in $\Pi_i \times \{v\}$ must be dominated by vertices in Q_v since $\Pi_i \times \{v\} \nsubseteq D$ and V_2 is a dominating set of graph $G\Box H - V_1$. Therefore $(i, v) \in L_i$. This implies that $|L_i| \geq |V(H) - N_H[P_i]|$. Hence

$$N \geq \sum_{i=1}^{\gamma(G)} |V(H) - N_H[P_i]|$$

Now it follows from (1) that

$$\begin{aligned} N &\geq \gamma(G)\gamma(H) - \sum_{i=1}^{\gamma(G)} |P_i| \\ &\geq \gamma(G)\gamma(H) - \sum_{i=1}^{\gamma(G)} |D_i|. \end{aligned}$$

So we obtain the following lower bound for N.

$$N \geq \gamma(G)\gamma(H) - |D| = \gamma(G)\gamma(H) - |V_1| - |V_2|. \tag{2}$$

For each $v \in V(H)$, $|R_v| \leq |Q_v|$. If it is not true, then

$$\{\, u \mid (u,v) \in Q_v\} \cup \{\, u_j \mid (j,v) \notin R_v\}$$

is a dominating set of G with cardinality

$$|Q_v| + (\gamma(G) - |R_v|) = \gamma(G) - (|R_v| - |Q_v|) < \gamma(G),$$

and we have a contradiction. This observation shows a upper bound for N.

$$N = \sum_{v \in V(H)} |R_v| \leq \sum_{v \in V(H)} |Q_v| = |V_2|. \tag{3}$$

It follows from (2) and (3) that

$$\gamma(G)\gamma(H) - |V_1| - |V_2| \leq N \leq |V_2|,$$

So we get $\gamma(G)\gamma(H) \leq |V_1| + 2|V_2| = \gamma_R(G\Box H)$. $\Box$

Remark: One may wonder if there is a similar result on Roman domination number as Vizing's conjecture. In fact, there are examples showing the inequality $\gamma_R(G)\gamma_R(H) \leq \gamma_R(G\Box H)$ fails, e.g., $\gamma_R(K_2) = 2$, but $\gamma_R(K_2\Box K_2) = 3$.

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